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Abstract. We investigate finite volume schemes for the one-dimensional Kerr-Debye model of electromagnetic propagation in nonlinear media. In this relaxation quasilinear hyperbolic system, the relaxation parameter is the response time of the media. When it tends to zero, the relaxed limit is known as the Kerr system. We show that basic explicit splitting methods fail to preserve this asymptotic. Following two different viewpoints, we construct splitting implicit and well-balanced explicit approximations which are stable, entropic and own the correct asymptotic behavior. Various numerical experiments are performed.

1. Introduction

Nonlinear Maxwell's equations are used for modeling nonlinear optical phenomena. For a nonlinear Kerr medium, the electromagnetic field (E, H) is linked to the electric and magnetic displacements D and B by the constitutive relations:

$$\begin{cases} B &= \mu_0 H \\ D &= \epsilon_0 E + P \end{cases}$$

where P is the nonlinear polarization.

If the medium exhibits an instantaneous response we have a Kerr model:

$$P = P_K = \epsilon_0 \epsilon_r |E|^2 E.$$

If the medium exhibits a finite response time τ we have a Kerr-Debye model:

$$P = P_{KD} = \epsilon_0 \chi E, \quad \partial_t \chi + \frac{1}{\tau}\chi = \frac{1}{\tau}\epsilon_r |E|^2.$$

See for example [22] or [24] for details.

So the Kerr-Debye model is a relaxation approximation of the Kerr model and τ is the relaxation parameter. Formally when τ tends to 0, χ converges to $\epsilon_r |E|^2$ and P_{KD} converges to P_K.

In the one-dimensional setting and after adimensionalization, we denote $(d(x,t), h(x,t))$, $((x,t) \in \mathbb{R} \times \mathbb{R}^+)$ the electromagnetic field. Moreover and as usual for relaxation systems, we denote ϵ the response time τ.

With those notations, writing Maxwell's equations for the Kerr model leads to the following Kerr system:

$$\begin{cases} \partial_t d + \partial_x h &= 0, \\ \partial_t h + \partial_x p(d) &= 0 \end{cases} \tag{1}$$

where p is the reciprocal function of

$$q(e) = e + e^3.$$

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The Kerr system is a p-system where the function p is strictly increasing and is strictly convex on $]-\infty, 0]$ and strictly concave on $[0, +\infty[$. It is strictly hyperbolic with the eigenvalues

$$\lambda_1(d) = -\sqrt{p'(d)} < 0 < \lambda_2(d) = \sqrt{p'(d)}\,. \tag{2}$$

The energy is a mathematical entropy for the Kerr system:

$$\mathcal{E}_K(d,h) = \frac{p(d)^2}{2} + \frac{h^2}{2} + \frac{3p(d)^4}{4} \tag{3}$$

and the entropy flux is given by

$$\mathcal{Q}_K(d,h) = h\,p(d). \tag{4}$$

On another hand, the Kerr-Debye model writes:

$$\begin{cases} \partial_t d_\epsilon + \partial_x h_\epsilon = 0, \\ \partial_t h_\epsilon + \partial_x \dfrac{d_\epsilon}{1+\chi_\epsilon} = 0, \\ \partial_t \chi_\epsilon = \dfrac{1}{\epsilon}\left[\left(\dfrac{d_\epsilon}{1+\chi_\epsilon}\right)^2 - \chi_\epsilon\right]. \end{cases} \tag{5}$$

In the following we denote

$$G(d,h,\chi) = \left(\frac{d}{1+\chi}\right)^2 - \chi\,. \tag{6}$$

In the domain $\Omega = \{(d,h,\chi),\ \chi \geq 0\}$ this system is strictly hyperbolic with the eigenvalues:

$$\lambda_1 = -\frac{1}{\sqrt{1+\chi_\epsilon}} < \lambda_2 = 0 < \lambda_3 = \frac{1}{\sqrt{1+\chi_\epsilon}}\,. \tag{7}$$

It is easy to see that all the three characteristic fields are linearly degenerate.

The equilibrium set for system (5) is given by

$$E = \left\{(d,h,\chi) \in \mathbb{R}^2 \times [0,+\infty[,\ \chi = \left(\frac{d}{1+\chi}\right)^2\right\} \tag{8}$$

which can also be written as

$$E = \left\{(d,h,\chi) \in \mathbb{R}^2 \times [0,+\infty[,\ \chi = p(d)^2\right\}. \tag{9}$$

In the smooth case, the convergence of the solutions of the initial value problem for the relaxation system (5) to those of the initial value problem for the relaxed system (1) has been proved in [16]. Convergence of smooth solutions also holds for the initial boundary value problem [8]. Global existence of smooth solutions holds for Kerr-Debye system, while shock creation can be proven for Kerr equations [9].

As far as one is concerned with weak solutions of Kerr system, to our knowledge there is no general convergence result. We are able to construct the general solution of the Riemann problem and we know that for a given shock wave, there exists a relaxation Kerr-Debye shock profile approximating it [3]. The energy density

$$\mathcal{E}_{KD}(d,h,\chi) = \frac{d^2}{2(1+\chi)} + \frac{h^2}{2} + \frac{\chi^2}{4} \tag{10}$$

is a strictly convex entropy for system (5) and the entropy flux is given by

$$\mathcal{Q}_{KD}(d,h,\chi) = \frac{h\,d}{1+\chi}\,. \tag{11}$$

For smooth solutions of system (5) we have the entropy dissipation property:

$$\partial_t \mathcal{E}_{KD}(d,h,\chi) + \partial_x \mathcal{Q}_{KD}(d,h,\chi) = \frac{-1}{2\epsilon} G(d,h,\chi)^2. \tag{12}$$

In this article, we construct numerical schemes for the Kerr-Debye system (5). Such approximations have to be accurate, stable and they must be asymptotic preserving: when ϵ tends to zero, we have to obtain a scheme which is consistent with the Kerr system (1).

The plan of the paper is the following: in Sect. 2 we study splitting schemes for system (5). We show that an explicit treatment of the source term is not efficient to insure the good asymptotic behavior. We then derive an implicit discretization of the source term and we prove the convergence of Newton's method to the unique solution of the scheme. From a practical point of view, this convergence is reached with very few iterations. Next, we construct a scheme based on the knowledge of the exact solution of the ODE

$$\chi' = \frac{1}{\epsilon} G(d,h,\chi)$$

when d is a constant. This approximation is also implicit and the unique solution of the scheme is computed by a dichotomy method. Both implicit schemes own the same relaxed limit which is an explicit consistent approximation of Kerr system. We prove that the positivity of χ is preserved and that a discrete entropy inequality holds. In view of those results, we conclude that such an approach cannot give rise to explicit asymptotic preserving schemes. We therefore turn our attention to explicit well-balanced schemes: it is the purpose of Sect. 3. Involving recent works issuing from numerical approximations of shallow-water equations [7, 13] or radiative transfer [6], we propose a relaxation scheme [2, 4, 5, 7, 18] such that the stiff source term enters the definition of the associated relaxation approximate Riemann solver. From this technique, we exhibit a relevant discrete form of the stiff source term which is next applied to a Godunov type scheme. Concerning this last numerical method, the positivity of χ is preserved and a discrete entropy inequality is established. In Sect. 4 we present numerical experiments: we explore the behavior of our schemes for large values of ϵ, and then we study their asymptotic behavior when ϵ tends to zero. In particular, we analyze relaxation shock profiles and solutions of the Riemann problem for Kerr system (1).

Throughout the paper we denote $u = (d,h,\chi)$, Δx the uniform space step, Δt the possibly variable time step, $t_0 = 0$, $t_n = t_{n-1} + \Delta t$,

$$x_{i-\frac{1}{2}} = (i - \frac{1}{2})\Delta x, \;\; C_i =]x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}[, \;\; i \in \mathbb{Z}$$

and

$$x_i = \frac{x_{i-\frac{1}{2}} + x_{i+\frac{1}{2}}}{2}.$$

The approximation of u on C_i at time t_n is denoted u_i^n.

We denote u_0 the initial data:

$$u(x,0) = u_0(x), \;\; x \in \mathbb{R}. \tag{13}$$

The numerical initial condition is in general taken as

$$u_i^0 = \int_{C_i} u_0(x)dx, \;\; i \in \mathbb{Z}. \tag{14}$$

However, if we take data in the equilibrium set E, we put

$$d_i^0 = \int_{C_i} d_0(x)dx, \;\; h_i^0 = \int_{C_i} h_0(x)dx, \;\; \chi_i^0 = p(d_i^0)^2, \;\; i \in \mathbb{Z}. \tag{15}$$

In all the sequel we use the following notation:

$$e = \frac{d}{1+\chi}\,. \tag{16}$$

2. Splitting techniques.

The first idea is to split the system (5) into a system of conservation laws and an ordinary differential system:

$$\begin{cases} \partial_t d_\epsilon + \partial_x h_\epsilon = 0, \\ \partial_t h_\epsilon + \partial_x \dfrac{d_\epsilon}{1+\chi_\epsilon} = 0, \\ \partial_t \chi_\epsilon = 0 \end{cases} \tag{17}$$

and

$$\begin{cases} \partial_t d_\epsilon = 0, \\ \partial_t h_\epsilon = 0, \\ \partial_t \chi_\epsilon = \dfrac{1}{\epsilon} G(u_\epsilon). \end{cases} \tag{18}$$

Suppose that the approximate solution u^n at time t_n is known. As all the three characteristic fields are linearly degenerate it is easy to approximate the first system (17) by Godunov's scheme and to obtain an intermediate solution $u^{n+\frac{1}{2}}$ at time t_{n+1}.

The second step consists in solving system (18) on $[t_n, t_{n+1}]$ with data $u^{n+\frac{1}{2}}$ at time t_n. At this stage several schemes can be constructed.

Let us detail the first step. Let u_h be a solution of system (17) on $\mathbb{R} \times [t_n, t_{n+1}]$ with

$$u_h(x, t_n) = u_i^n \text{ if } x \in C_i\,, \quad i \in \mathbb{Z}. \tag{19}$$

It is well-known that for a half CFL condition one can construct u_h by juxtaposing solutions of Riemann problems at cell interfaces. It is therefore enough to give the solution of the Riemann problem for system (17). Let u_0 be the initial data defined by

$$u(x,0) = u_0(x) = \left| \begin{array}{lll} u_- & \text{if} & x < 0, \\ u_+ & \text{if} & x > 0 \end{array} \right. \tag{20}$$

where u_- and u_+ are constant states. As all the characteristic fields are linearly degenerate the solution of problem (17)(20) consists in three contact discontinuities propagating at characteristic velocities:

$$u(x,t) = \left| \begin{array}{lll} u_- & \text{if} & \dfrac{x}{t} < \dfrac{-1}{\sqrt{1+\chi_-}} \\ u_1 & \text{if} & \dfrac{-1}{\sqrt{1+\chi_-}} < \dfrac{x}{t} < 0 \\ u_2 & \text{if} & 0 < \dfrac{x}{t} < \dfrac{+1}{\sqrt{1+\chi_+}} \\ u_+ & \text{if} & \dfrac{x}{t} > \dfrac{+1}{\sqrt{1+\chi_+}} \end{array} \right. \tag{21}$$

The intermediate constant states u_1 and u_2 are calculated *via* the Rankine-Hugoniot jump conditions. We find

$$\left\{\begin{aligned} & h_1 = h_2 = h^\star = \frac{-(e_+ - e_-)\sqrt{1+\chi_-}\sqrt{1+\chi_+} + h_+\sqrt{1+\chi_-} + h_-\sqrt{1+\chi_+}}{\sqrt{1+\chi_+}+\sqrt{1+\chi_-}} \\ & d_1 = d_- - (h^\star - h_-)\sqrt{1+\chi_-} \\ & d_2 = d_+ - (h_+ - h^\star)\sqrt{1+\chi_+} \\ & \chi_1 = \chi_- \\ & \chi_2 = \chi_+ \end{aligned}\right. \tag{22}$$

The numerical flux of the Godunov scheme is then given by

$$\left\{\begin{aligned} & h_{i+\frac12} = \frac{h_i\sqrt{1+\chi_{i+1}} + h_{i+1}\sqrt{1+\chi_i} - (e_{i+1}-e_i)\sqrt{1+\chi_i}\sqrt{1+\chi_{i+1}}}{\sqrt{1+\chi_{i+1}}+\sqrt{1+\chi_i}} \\ & e_{i+\frac12} = \frac{h_i - h_{i+1} + e_i\sqrt{1+\chi_i} + e_{i+1}\sqrt{1+\chi_{i+1}}}{\sqrt{1+\chi_{i+1}}+\sqrt{1+\chi_i}} \end{aligned}\right. \tag{23}$$

and we put

$$u_i^{n+\frac12} = u_i^n - \frac{\Delta t}{\Delta x}(F^n_{i+\frac12} - F^n_{i-\frac12}), \qquad F^n_{i+\frac12} = {}^t(h^n_{i+\frac12}, e^n_{i+\frac12}, 0) \qquad i \in \mathbb{Z}. \tag{24}$$

The following lemma is useful in the sequel.

Lemma 1. *For $i \in \mathbb{Z}$ and $n \geq 0$ let $u^n_{h,i+\frac12}(x,t)$ be the solution of the Riemann problem for system (17) with data*

$$u(x,t_n) = \left| \begin{array}{lll} u_i^n & \text{if} & x < x_{i+\frac12}\,, \\ u_{i+1}^n & \text{if} & x > x_{i+\frac12}\,. \end{array}\right. \tag{25}$$

Under the half CFL condition

$$\sup_{i\in\mathbb{Z}} \frac{\Delta t}{\Delta x\sqrt{1+\chi_i^n}} \leq \frac12 \tag{26}$$

the function u_h defined by

$$u_h(x,t) = u^n_{h,i+\frac12}(x,t) \ \text{ if } \ x \in]x_i, x_{i+1}[$$

is a solution of problem (17)(19) on $\mathbb{R}\times[t_n,t_{n+1}]$ and we have

$$u_i^{n+\frac12} = \frac{1}{\Delta x}\int_{C_i} u_h(x,t_{n+1})dx.$$

Proof. This result is classical, see [14] for example, so we omit the proof. □

We now turn our attention to system (18). For every cell C_i, we take $u_i^{n+\frac12}$, defined in (24), as data for $t = t_n$. Then, u_i^{n+1} is an approximate solution of (18) at time t_{n+1}. The two first equations give

$$d_i^{n+1} = d_i^{n+\frac12}, \ \ h_i^{n+1} = h_i^{n+\frac12}\,. \tag{27}$$

2.1. **Explicit schemes.** We observe that the solution of the third equation can be expressed as

$$\chi(x,t) = \mathrm{e}^{-(t-t_n)/\epsilon}\chi(x,t_n) + \frac{1}{\epsilon}\int_{t_n}^{t} \mathrm{e}^{-(t-s)/\epsilon} e^2(x,s)ds\,, \tag{28}$$

where e is defined by (16). On the cell C_i, we approach this formula by replacing e by

$$e_i^{n+\frac{1}{2}} = \frac{d_i^{n+\frac{1}{2}}}{1+\chi_i^{n+\frac{1}{2}}}$$

and by taking the average on C_i. As $\chi_i^{n+\frac{1}{2}} = \chi_i^n$, we obtain for $t = t_{n+1}$:

$$\chi_i^{n+1} = \mathrm{e}^{-\Delta t/\epsilon}\chi_i^n + (1-\mathrm{e}^{-\Delta t/\epsilon})\left[\frac{d_i^{n+1}}{1+\chi_i^n}\right]^2 . \tag{29}$$

A second approximation is based on the fact that for a one-half CFL condition, the solution of system (17) with data u^n is explicitly known in the domain $C_i\times[t_n,t_{n+1}]$. In particular, we have for all $(x,s)\in C_i\times[t_n,t_{n+1}]$:

$$e(x,s) = \left|\begin{array}{lll} e^n_{i-\frac{1}{2}} & \text{if} & x_{i-\frac{1}{2}} < x < x_{i-\frac{1}{2}} + \dfrac{s-t_n}{\sqrt{1+\chi_i^n}}, \\ e^n_i & \text{if} & x_{i-\frac{1}{2}} + \dfrac{s-t_n}{\sqrt{1+\chi_i^n}} < x < x_{i+\frac{1}{2}} - \dfrac{s-t_n}{\sqrt{1+\chi_i^n}}, \\ e^n_{i+\frac{1}{2}} & \text{if} & x_{i+\frac{1}{2}} - \dfrac{s-t_n}{\sqrt{1+\chi_i^n}} < x < x_{i+\frac{1}{2}} \end{array}\right. \tag{30}$$

where $e^n_{i\pm\frac{1}{2}}$ are defined in (23). We replace $e(x,s)$ by this value in (28) and take the average of the result over the cell C_i. We obtain:

$$\begin{aligned} \chi_i^{n+1} &= \mathrm{e}^{-\Delta t/\epsilon}\chi_i^n + (1-\mathrm{e}^{-\Delta t/\epsilon})(e_i^n)^2 \\ &\quad + \frac{\Delta t - \epsilon(1-\mathrm{e}^{-\Delta t/\epsilon})}{\Delta x\sqrt{1+\chi_i^n}}\left[(e^n_{i-\frac{1}{2}})^2 - 2(e_i^n)^2 + (e^n_{i+\frac{1}{2}})^2\right]. \end{aligned} \tag{31}$$

2.2. **The formal limit of the explicit splitting schemes.** Let us let ϵ tend to zero in the two schemes above. The first (Godunov) part of each of them is unchanged. The value of χ in formula (29) tends to

$$\chi_i^{n+1} = \left[\frac{d_i^{n+1}}{1+\chi_i^n}\right]^2 .$$

The one in formula (31) tends to

$$\chi_i^{n+1} = (e_i^n)^2 + \frac{\Delta t}{\Delta x\sqrt{1+\chi_i^n}}\left[(e^n_{i-\frac{1}{2}})^2 - 2(e_i^n)^2 + (e^n_{i+\frac{1}{2}})^2\right] .$$

We observe that in neither of the two cases the equilibrium is not reached. The numerical experiments show in fact that the asymptotic behavior of those schemes is sometimes false. In practice the scheme with (31) gave almost the same results than the one with (29), when ϵ is large as well as in the relaxation limit $\epsilon \to 0$. Therefore we do not consider anymore the scheme with (31) in this paper.

2.3. **An implicit scheme.** We again consider formula (28) but here we replace e by

$$e_i^{n+1} = \frac{d_i^{n+1}}{1+\chi_i^{n+1}}$$

and we obtain:

$$\chi_i^{n+1} = \mathrm{e}^{-\Delta t/\epsilon}\chi_i^n + (1-\mathrm{e}^{-\Delta t/\epsilon})\left[\frac{d_i^{n+1}}{1+\chi_i^{n+1}}\right]^2. \tag{32}$$

We then use Newton's method to find χ_i^{n+1}.

Lemma 2. *Let d_i^{n+1} and $\chi_i^n \geq 0$ be fixed. The equation (32) owns a unique solution $\chi_i^{n+1} \geq 0$ and Newton's method converges to this solution when $\chi_i^n \mathrm{e}^{-\Delta t/\epsilon}$ is taken as initial value.*

Proof. We suppose that $\chi_i^n \geq 0$. Let us set $X = 1 + \chi_i^{n+1}$. Equation (32) can be written as $f(X) = 0$ where

$$f(X) = X^3 - X^2(1+\chi_i^n \mathrm{e}^{-\Delta t/\epsilon}) - (d_i^{n+1})^2(1-\mathrm{e}^{-\Delta t/\epsilon}).$$

We observe that this function has a negative maximum for $X = 0$, a negative minimum for $X = \frac{2}{3}(1+\chi_i^n\mathrm{e}^{-\Delta t/\epsilon})$ and that it is strictly convex and increasing on the interval $[\frac{2}{3}(1+\chi_i^n\mathrm{e}^{-\Delta t/\epsilon}), +\infty[$. Moreover $f(1+\chi_i^n\mathrm{e}^{-\Delta t/\epsilon})$ is negative. Hence the unique root of f is greater than $1+\chi_i^n\mathrm{e}^{-\Delta t/\epsilon}$ and Newton's method converges when $1+\chi_i^n\mathrm{e}^{-\Delta t/\epsilon}$ is taken as initial value. □

2.4. **The formal limit of the implicit splitting scheme.** Let us let ϵ tend to zero in the implicit scheme above. The first (Godunov) part is unchanged. The value of χ in formula (32) tends to

$$\chi_i^{n+1} = \left[\frac{d_i^{n+1}}{1+\chi_i^{n+1}}\right]^2.$$

The equilibrium is reached: this scheme is asymptotic preserving. Let us observe that in the limit:

$$\chi_i^n = p(d_i^n)^2 = (\mathbf{e}_i^n)^2$$

where we denote

$$\mathbf{e} = p(d). \tag{33}$$

The relaxed scheme for the Kerr system is given by the following consistent numerical flux functions:

$$\begin{cases} \mathbf{h}_{i+\frac{1}{2}} = \dfrac{h_i\sqrt{1+\mathbf{e}_{i+1}^2} + h_{i+1}\sqrt{1+\mathbf{e}_i^2} - (\mathbf{e}_{i+1}-\mathbf{e}_i)\sqrt{1+\mathbf{e}_i^2}\sqrt{1+\mathbf{e}_{i+1}^2}}{\sqrt{1+\mathbf{e}_{i+1}^2}+\sqrt{1+\mathbf{e}_i^2}} \\[2ex] \mathbf{e}_{i+\frac{1}{2}} = \dfrac{h_i - h_{i+1} + \mathbf{e}_i\sqrt{1+\mathbf{e}_i^2} + \mathbf{e}_{i+1}\sqrt{1+\mathbf{e}_{i+1}^2}}{\sqrt{1+\mathbf{e}_{i+1}^2}+\sqrt{1+\mathbf{e}_i^2}} \end{cases} \tag{34}$$

We denote $\overline{u} = {}^t(d,h)$. The relaxed scheme can be written as

$$\overline{u}_i^{n+1} = \overline{u}_i^n - \frac{\Delta t}{\Delta x}(\overline{F}_{i+\frac{1}{2}}^n - \overline{F}_{i-\frac{1}{2}}^n), \qquad \overline{F}_{i+\frac{1}{2}}^n = {}^t(\mathbf{h}_{i+\frac{1}{2}}^n, \mathbf{e}_{i+\frac{1}{2}}^n), \qquad i \in \mathbb{Z}. \tag{35}$$

2.5. **Exact solving of the χ equation.** Here, $u^{n+\frac{1}{2}}$ is still given by (23)(24), we take (d^{n+1}, h^{n+1}) as (27) and now for each cell C_i, we solve exactly the third equation of system (18). Hence we look for the solution of the initial value problem

$$\left\{\begin{array}{l} \chi' = \dfrac{1}{\epsilon}\left[\left(\dfrac{d}{1+\chi}\right)^2 - \chi\right], \quad t > t_0 \\ \\ \chi(t_0) = \chi_0 \end{array}\right. \tag{36}$$

where d is a real constant and χ_0 is nonnegative. As already observed, the right-hand side is zero if and only if

$$\chi = \mathbf{e}^2$$

with the notation (33).

Lemma 3. *For all $\chi_0 \geq 0$ the problem (36) has a unique global solution $\chi \in C^1([t_0, +\infty[)$.*

If $\chi_0 = \mathbf{e}^2$ then the solution of problem (36) is χ_0 for all times.

If $0 \leq \chi_0 < \mathbf{e}^2$ then χ is an increasing function and

$$\forall t > t_0 \quad \chi_0 < \chi(t) < \mathbf{e}^2 .$$

If $\chi_0 > \mathbf{e}^2$ then χ is a decreasing function and

$$\forall t > t_0 \quad \mathbf{e}^2 < \chi(t) < \chi_0 .$$

Proof. We study the sign of $\chi'(t)$, that is the sign of the function φ defined for $y \geq 0$ by

$$\varphi(y) = d^2 - y(1+y)^2.$$

We have $d = q(\mathbf{e})$, so that

$$\varphi(y) = (\mathbf{e}^2 - y)\left[(\mathbf{e}^2+1)^2 + y(\mathbf{e}^2+2) + y^2\right].$$

Therefore, $\varphi(y) > 0$ if and only if $\mathbf{e}^2 > y$. The conclusion follows by the general theory of ODE's. ☐

The second lemma gives the solution of the equation:

Lemma 4. *Let $\chi_0 \neq \mathbf{e}^2$ be a nonnegative real number. Let Ψ be the function defined for $0 \leq y \neq \mathbf{e}^2$ by*

$$\begin{aligned} \Psi(y) \quad = \quad & -\frac{\mathbf{e}^2+1}{3\mathbf{e}^2+1}\ln|y-\mathbf{e}^2| - \frac{\mathbf{e}^2}{3\mathbf{e}^2+1}\ln\left[\frac{4(y^2+y(\mathbf{e}^2+2)+(\mathbf{e}^2+1)^2)}{\mathbf{e}^2(3\mathbf{e}^2+4)}\right] \\ & -\frac{2\mathbf{e}}{(3\mathbf{e}^2+1)\sqrt{3\mathbf{e}^2+4}}\mathrm{Arctg}\left[\frac{2y+\mathbf{e}^2+2}{\mathbf{e}(3\mathbf{e}^2+4)}\right]. \end{aligned} \tag{37}$$

The function Ψ is strictly convex, increasing on the interval $[0, \mathbf{e}^2[$, decreasing on the interval $]\mathbf{e}^2, +\infty[$ and

$$\lim_{y \to \mathbf{e}^2} \Psi(y) = +\infty.$$

Moreover the solution of problem (36) is

$$\chi(t) = \Psi^{-1}\left(\Psi(\chi_0) + \frac{t-t_0}{\epsilon}\right) \quad t \geq t_0 \tag{38}$$

where Ψ^{-1} is the reciprocal function of Ψ on $[0, \mathbf{e}^2[$ if $\chi_0 < \mathbf{e}^2$, on $]\mathbf{e}^2, +\infty[$ if $\chi_0 > \mathbf{e}^2$.

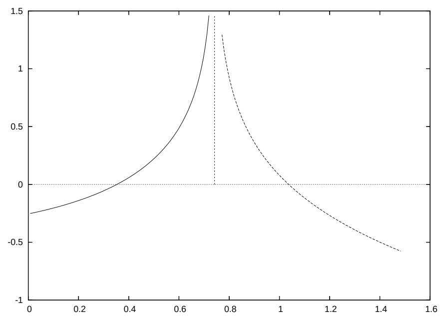

FIGURE 1. Representation of the function Ψ.

Proof. We know that for $\chi_0 \neq \mathbf{e}^2$ the solution $\chi(t)$ satisfies the following equation:

$$\int_{\chi_0}^{\chi(t)} \frac{(1+\chi)^2}{d^2 - \chi(1+\chi)^2} d\chi = \frac{t - t_0}{\epsilon} .$$

We then proceed classically and find Ψ as a primitive of the rational function in the left-hand side. The properties of Ψ are easy to verify. □

We do not know Ψ^{-1} explicitly so that this scheme is implicit. We use a dichotomy method. In the following, we do not take into account this numerical approximation and we analyze the splitting method with the exact value

$$\chi_i^{n+1} = (\Psi_i^{n+1})^{-1} \left(\Psi_i^{n+1}(\chi_i^n) + \frac{\Delta t}{\epsilon} \right), \quad i \in \mathbb{Z}. \tag{39}$$

Here we denote Ψ_i^{n+1} the function Ψ with $d = d_i^{n+1}$.

2.6. **Formal limit of the scheme.** When ϵ tends to zero, then χ_i^{n+1} computed by formula (39) tends to $(\mathbf{e}_i^{n+1})^2$. Therefore, the scheme is asymptotic preserving and the relaxed scheme for Kerr system is the same as the one we found with the implicit scheme, that is (35) with (34).

2.7. **Stability properties.** We remark that the positivity of χ is preserved by our schemes:

Proposition 1. *We consider the schemes (23) (24) with explicit (29), implicit (32), or (39) as second step. We impose the CFL condition (26). If the initial data χ_0 is nonnegative then, for all $n \geq 0$ and for all $i \in \mathbb{Z}$ the value of χ_i^n is nonnegative.*

Proof. First, we remark that $\chi_i^0 \geq 0$ for all $i \in \mathbb{Z}$. Then, Godunov's scheme is such that

$$\chi_i^{n+\frac{1}{2}} = \chi_i^n .$$

From formulas (29), (32) and lemma 3, the three considered discretizations of system (18) preserve the positivity of χ, which ends the proof. □

We now turn our attention to entropy properties. We begin with the following result, due to the linear degeneracy of the fields:

Lemma 5. *Let $(\mathcal{E}, \mathcal{Q})$ be an entropy-entropy flux pair for system (17). If the left and right states u_l and u_r are connected by a i-contact discontinuity propagating with velocity $\lambda_i(u_l) = \lambda_i(u_r)$, then we have the entropy equality*

$$\lambda_i(u_l)\,[\mathcal{E}(u_r) - \mathcal{E}(u_l)] = \mathcal{Q}(u_r) - \mathcal{Q}(u_l). \tag{40}$$

As a consequence, if u_l and u_r are connected by a 2-contact discontinuity then $\mathcal{Q}(u_l) = \mathcal{Q}(u_r)$.

Proof. This result is classical and can be obtained by using the fact that u_l and u_r are connected by a i-contact discontinuity if and only if u_l and u_r belong to the same integral curve of i-right eigenvectors, see for example [21]. □

For the solution of the Riemann problem (17)(25) we denote

$$\left\{\begin{array}{l} u^{n,1}_{h,i+\frac12} = u^n_{h,i+\frac12}(x,t) \ \text{ if } \ -\dfrac{1}{\sqrt{1+\chi^n_i}} < \dfrac{x-x_{i+\frac12}}{t-t_n} < 0, \\ \\ u^{n,2}_{h,i+\frac12} = u^n_{h,i+\frac12}(x,t) \ \text{ if } \ 0 < \dfrac{x-x_{i+\frac12}}{t-t_n} < \dfrac{1}{\sqrt{1+\chi^n_{i+1}}}. \end{array}\right. \tag{41}$$

As $u^{n,1}_{h,i+\frac12}$ and $u^{n,2}_{h,i+\frac12}$ are connected by a 2-contact discontinuity, by lemma 5 we have $\mathcal{Q}(u^{n,1}_{h,i+\frac12}) = \mathcal{Q}(u^{n,2}_{h,i+\frac12})$ for all entropy flux $\mathcal{Q}$. We denote this common value by

$$Q^n_{i+\frac12} = \mathbf{Q}(u^n_i, u^n_{i+1}) = \mathcal{Q}(u^{n,1}_{h,i+\frac12}) = \mathcal{Q}(u^{n,2}_{h,i+\frac12}). \tag{42}$$

We are now in position to prove the cell entropy inequality for the conservation law part of the scheme:

Proposition 2. *We consider the scheme (24) with (23) and the CFL condition (26). Let $(\mathcal{E},\mathcal{Q})$ be an entropy-entropy flux pair for system (17). The function $\mathbf{Q}$ defined in (42) is a numerical entropy flux and we have the discrete cell entropy inequality:*

$$\frac{\mathcal{E}(u_i^{n+\frac12}) - \mathcal{E}(u_i^n)}{\Delta t} + \frac{Q^n_{i+\frac12} - Q^n_{i-\frac12}}{\Delta x} \le 0. \tag{43}$$

Proof. This result is rather classical, we give the proof for the sake of completeness. We denote

$$\lambda = \frac{1}{\sqrt{1+\chi^n_i}}.$$

By lemma 1 and Jensen's inequality:

$$\begin{aligned} \mathcal{E}(u_i^{n+\frac12}) \quad &\le \quad \frac{1}{\Delta x}\int_{x_{i-\frac12}}^{x_{i-\frac12}+\lambda\Delta t} \mathcal{E}(u^{n,2}_{h,i-\frac12})dx \\ &\quad + \frac{1}{\Delta x}\int_{x_{i-\frac12}+\lambda\Delta t}^{x_{i+\frac12}-\lambda\Delta t} \mathcal{E}(u^n_i)dx + \frac{1}{\Delta x}\int_{x_{i+\frac12}-\lambda\Delta t}^{x_{i+\frac12}} \mathcal{E}(u^{n,1}_{h,i+\frac12})dx. \end{aligned}$$

This can be also written as

$$\mathcal{E}(u_i^{n+\frac12}) \quad \le \quad \mathcal{E}(u^n_i) - \lambda\frac{\Delta t}{\Delta x}\left[-\mathcal{E}(u^{n,2}_{h,i-\frac12}) + 2\mathcal{E}(u^n_i) - \mathcal{E}(u^{n,1}_{h,i+\frac12})\right].$$

By (40):

$$\lambda\left[\mathcal{E}(u^n_i) - \mathcal{E}(u^{n,2}_{h,i-\frac12})\right] = \mathcal{Q}(u^n_i) - \mathcal{Q}(u^{n,2}_{h,i-\frac12}) \quad (3-\text{contact discontinuity})$$

and

$$-\lambda\left[\mathcal{E}(u^{n,1}_{h,i+\frac12}) - \mathcal{E}(u^n_i)\right] = \mathcal{Q}(u^{n,1}_{h,i+\frac12}) - \mathcal{Q}(u^n_i) \quad (1-\text{contact discontinuity}),$$

which ends the proof. □

We now take into account the source term to obtain a discrete version of equality (12) for the implicit and exact solving based schemes.

Theorem 1. *We consider the schemes (24)(32) and (24)(39), with (23) and the CFL condition (26). Let* $\mathbf{Q}_{KD}(u_i, u_{i+1}) = Q_{KD,i+\frac{1}{2}}$ *be the numerical entropy flux function (42) for the entropy flux* $\mathcal{Q}_{KD}$*. The following cell entropy dissipation holds for both schemes:*

$$\frac{\mathcal{E}_{KD}(u_i^{n+1}) - \mathcal{E}_{KD}(u_i^n)}{\Delta t} + \frac{Q^n_{KD,i+\frac{1}{2}} - Q^n_{KD,i-\frac{1}{2}}}{\Delta x} \leq -\frac{1}{2\epsilon} G(u_i^{n+1})^2 \,. \tag{44}$$

Proof. We first consider the implicit scheme (24)(32). One can see u_i^{n+1} as the exact solution at time t_{n+1} of

$$\begin{cases} \partial_t d = 0, \\ \partial_t h = 0, \\ \partial_t y = \dfrac{1}{\epsilon}\left[\left(\dfrac{d}{1+\chi_i^{n+1}}\right)^2 - y\right] \end{cases} \tag{45}$$

with data $u_i^{n+\frac{1}{2}}$ at time t_n. Let us denote $v_i(t) = (d_i^{n+1}, h_i^{n+1}, y_i(t))$ the solution of this same problem for $t \in [t_n, t_{n+1}]$:

$$v_i(t_{n+1}) = u_i^{n+1} \,.$$

We multiply the system (45) by $\mathcal{E}'_{KD}(v_i)$. We obtain

$$\partial_t \mathcal{E}_{KD}(v_i) = -\frac{1}{2\epsilon}\left[\left(\frac{d_i^{n+1}}{1+\chi_i^{n+1}}\right)^2 - y_i(t)\right] G(v_i(t)) \,.$$

Hence

$$\mathcal{E}_{KD}(u_i^{n+1}) = \mathcal{E}_{KD}(u_i^{n+\frac{1}{2}}) - \frac{1}{2}\int_{t_n}^{t_{n+1}} G(v_i)\, y_i' \, dt.$$

Let us denote $g(t)$ the function inside the integral. As $y_i'' = -\frac{1}{\epsilon} y_i'$ we have

$$g' = -\frac{1}{\epsilon} y_i' G(v_i) + (y_i')^2 \partial_3 G(v_i)$$

The sign of y_i' is constant on $[t_n, t_{n+1}]$ because

$$y_i'(t) = \frac{1}{\epsilon}\, e^{-\frac{t-t_n}{\epsilon}} \left[\left(\frac{d_i^{n+1}}{1+\chi_i^{n+1}}\right)^2 - \chi_i^n\right].$$

Moreover, $\partial_3 G < 0$ for $y > 0$ and $G(v_i(t_{n+1})) = \epsilon\, y_i'(t_{n+1})$. Therefore, if $y_i' \geq 0$ then $\chi_i^n \leq y_i(t) \leq \chi_i^{n+1}$ and $0 \leq G(u_i^{n+1}) \leq G(v_i(t))$. If $y_i' \leq 0$ then $\chi_i^n \geq y_i(t) \geq \chi_i^{n+1}$ and $0 \geq G(u_i^{n+1}) \geq G(v_i(t))$. In each case we have $y_i' G(v_i) \geq 0$. Therefore, the function g is not increasing and

$$\mathcal{E}_{KD}(u_i^{n+1}) \leq \mathcal{E}_{KD}(u_i^{n+\frac{1}{2}}) - \frac{\Delta t}{2\epsilon} G(u_i^{n+1})^2 \,. \tag{46}$$

The same inequality can be obtained with the scheme (24)(39): we recall that u_i^{n+1} is the exact solution at time t_{n+1} of system (18) with data $u_i^{n+\frac{1}{2}}$ at time t_n. Let us denote $u_i(t) = (d_i^{n+1}, h_i^{n+1}, \chi_i(t))$ the solution of this same problem for $t \in [t_n, t_{n+1}]$. We multiply the system (18) by $\mathcal{E}'_{KD}(u_i)$. We obtain

$$\partial_t \mathcal{E}_{KD}(u_i) = -\frac{1}{2\epsilon} G(u_i(t))^2 \,.$$

Hence

$$\mathcal{E}_{KD}(u_i^{n+1}) = \mathcal{E}_{KD}(u_i^{n+\frac{1}{2}}) - \frac{1}{2\epsilon}\int_{t_n}^{t_{n+1}} G(u_i(t))^2 dt.$$

Moreover

$$\frac{d}{dt}(G(u_i)^2) = 2\epsilon\, \chi_i'(t)^2\, \partial_3 G(u_i(t)) \leq 0$$

so that (46) holds. We then use inequality (43) and obtain the result. □

As a corollary, we have a cell entropy inequality for the Kerr relaxed scheme (35) with (34). Actually, $\mathcal{E}_{KD}$ is an entropy extension of the Kerr entropy $\mathcal{E}_K$ in the sense of [10]. Let us denote

$$\mathcal{P}\overline{u} = \begin{pmatrix} d \\ h \\ p(d)^2 \end{pmatrix}$$

the equilibrium state for $\overline{u} = {}^t(d,h)$. We have

$$\mathcal{E}_{KD}(\mathcal{P}\overline{u}) = \mathcal{E}_K(\overline{u}), \qquad \mathcal{Q}_{KD}(\mathcal{P}\overline{u}) = \mathcal{Q}_K(\overline{u})$$

We define the Kerr numerical entropy flux as

$$\mathbf{Q}_K(\overline{u},\overline{v}) = \mathbf{Q}_{KD}(\mathcal{P}\overline{u},\mathcal{P}\overline{v}), \qquad Q^n_{K,i+\frac{1}{2}} = \mathbf{Q}_K(\overline{u}^n_i,\overline{u}^n_{i+1}). \tag{47}$$

As the Kerr-Debye numerical entropy flux is consistent, we have

$$\mathbf{Q}_K(\overline{u},\overline{u}) = \mathcal{Q}_K(\overline{u}).$$

As for all $n \geq 1$ and $i \in \mathbb{Z}$, u^n_i is an equilibrium state, the inequality (43) gives

$$\mathcal{E}_{KD}(u_i^{n+\frac{1}{2}}) \leq \mathcal{E}_K(\overline{u}^n_i) - \frac{\Delta t}{\Delta x}(Q^n_{K,i+\frac{1}{2}} - Q^n_{K,i-\frac{1}{2}}).$$

The second step of the relaxed scheme is just a projection onto equilibrium which minimizes the entropy. More precisely, we have

$$\frac{\partial \mathcal{E}_{KD}}{\partial \chi}(\mathcal{P}\overline{u}) = 0, \qquad \frac{\partial^2 \mathcal{E}_{KD}}{\partial \chi^2}(\mathcal{P}\overline{u}) > 0.$$

As $d_i^{n+1} = d_i^{n+\frac{1}{2}}$, $h_i^{n+1} = h_i^{n+\frac{1}{2}}$, and $u_i^{n+1} = \mathcal{P}\overline{u}_i^{n+1}$ the following corollary holds:

Corollary 1. *We consider the relaxed scheme (35) with (34) and the CFL condition (26). Let $\mathbf{Q}_{K,i+\frac{1}{2}}$ be the numerical entropy flux function defined in (47). The following cell-entropy inequality holds:*

$$\frac{\mathcal{E}_K(\overline{u}_i^{n+1}) - \mathcal{E}_K(\overline{u}_i^n)}{\Delta t} + \frac{Q^n_{KD,i+\frac{1}{2}} - Q^n_{KD,i-\frac{1}{2}}}{\Delta x} \leq 0\,. \tag{48}$$

3. Well-balanced schemes

At the discrepancy with the above presented schemes, the present section concerns the derivation of approximate Riemann solver including, in a sense to be specified, the source term. After the pioneer work by Greenberg-LeRoux [15] (see also [7, 6, 13] to further extensions), we propose to introduce a new variable Z defined as follows: $Z(x,t) = x$. As a consequence, we have

$$\partial_t Z(x,t) = 0 \quad \text{and} \quad \partial_x Z(x,t) = 1.$$

Now, when considering an hyperbolic system with source terms in the form:

$$\partial_t w + \partial_x f(w) = \frac{1}{\epsilon} R(w), \tag{49}$$

the original idea consists in studying approximations for the following extended system:

$$\begin{cases} \partial_t w + \partial_x f(w) = \frac{1}{\epsilon} R(w) \partial_x Z, \\ \partial_t Z = 0 \end{cases} \tag{50}$$

Such an approach has been used with large benefits as soon as (49) coincides with shallow-water equations [7, 13] or radiative transfer equations [6] for instance. Unfortunately, when considering the Kerr-Debye model (5):

$$w = u_\epsilon, \quad f(u_\epsilon) = \begin{pmatrix} h_\epsilon \\ \dfrac{d_\epsilon}{1+\chi_\epsilon} \\ 0 \end{pmatrix}, \quad R(u_\epsilon) = \begin{pmatrix} 0 \\ 0 \\ \left(\dfrac{d_\epsilon}{1+\chi_\epsilon}\right)^2 - \chi_\epsilon \end{pmatrix}, \tag{51}$$

the extended system (50) is easily shown to be not hyperbolic. Indeed, the Jacobian flux matrix associated with (50)(51) admits four real eigenvalues but it is not diagonalisable in $\mathbb{R}$.

3.1. **A relaxation model.** To overcome the problems arising with weakly hyperbolic systems, we propose to approximate the weak solutions of (50)(51) by the weak solutions of a suitable first-order system with singular perturbations [10, 2, 20], a namely relaxation model. Following the work of Jin-Xin [18] (see also [7, 11, 4, 5] where several extensions are detailed), we suggest to consider the following nonlinear nonconservative relaxation model:

$$\begin{cases} \partial_t d + \partial_x h = 0, \\ \partial_t h + \partial_x \Pi = 0, \\ \partial_t \chi + \partial_x \Sigma = \dfrac{1}{\epsilon}(\Pi^2 - \chi)\partial_x Z, \\ \partial_t Z = 0, \end{cases} \tag{52}$$

$$\begin{cases} \partial_t \Pi + a^2 \partial_x h = \mu \left(\dfrac{d}{1+\chi} - \Pi \right), \\ \partial_t \Sigma + a^2 \partial_x \chi = -\mu \Sigma, \end{cases} \tag{53}$$

where $a > 0$ is a relaxation parameter to be defined and μ is a parameter devoted to tend to infinity.

The relaxation model (52)(53) involves two new variables, Π and Σ, coming with their own evolution laws (53). Clearly and at least formally, in the limit of μ to infinity, the pair (Π, Σ) tends to $(\frac{d}{1+\chi}, 0)$. This limit will be referred to as the equilibrium limit. As a consequence, as μ goes to infinity, we recover the initial system (50)(51) from the relaxation system (52).

Let us note from now on that the relaxation parameter a must satisfy additional stability conditions to prevent instabilities whenever μ goes to infinity. This condition, a Whitham sub-characteristic type restriction [23] (see also [2, 5, 7]), reads as follows:

$$a^2 > \frac{1}{1+\chi}. \tag{54}$$

In the present work, we do not prove this restriction but it will be adopted testing the resulting relaxation scheme.

For the sake of simplicity in the notations, let us introduce the relaxation state vector $U = {}^t({}^t u, Z, \Pi, \Sigma)$, associated with the admissible state space:

$$\mathcal{V} = \{U \in \mathbb{R}^6, \chi \geq 0\}.$$

The first result we give concerns the linear degeneracy of the fields of (52)(53). This property turns out to be essential to make easily solvable the associated Riemann problem.

Lemma 6. *Let be given $a > 0$ and assume $\mu = 0$. The first-order system $(52)(53)_{\mu=0}$ is hyperbolic for all $U \in \mathcal{V}$. The eigenvalues, defined by $\lambda^0 = 0$ and $\lambda^\pm = \pm a$ are double. The associated fields are linearly degenerate.*

The proof of this result turns out to be classic (for instance, see [14]) and we skip it. We complete the above lemma giving the Riemann invariants associated with each field. Performing an easy algebra analysis of $(52)(53)_{\mu=0}$, with clear notations the eigenvectors associated with each eigenvalue read as follows:

$$r^1_{\lambda^0} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad r^2_{\lambda^0} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ \frac{\Pi^2-\chi}{\epsilon} \end{pmatrix}, \quad r^1_{\lambda^\pm} = \begin{pmatrix} 1 \\ \pm a \\ 0 \\ 0 \\ a^2 \\ 0 \end{pmatrix}, \quad r^2_{\lambda^\pm} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ \pm a \end{pmatrix}.$$

Riemann invariants associated with λ^0 (respectively $\lambda^\pm$), denoted Φ_{λ^0} (resp. $\Phi_{\lambda^\pm}$), satisfy $\nabla_U \Phi . r = 0$ where $r = r^{1,2}_{\lambda^0}$ (resp. $r = r^{1,2}_{\lambda^\pm}$). We easily deduce

$$\begin{aligned} &\Phi^1_{\lambda^0}(U) = h, \quad \Phi^2_{\lambda^0}(U) = \Pi, \quad \Phi^3_{\lambda^0}(U) = \chi, \quad \Phi^4_{\lambda^0}(U) = \Sigma - \frac{\Pi^2 - \chi}{\epsilon} Z, \\ &\Phi^1_{\lambda^\pm}(U) = Z, \quad \Phi^2_{\lambda^\pm}(U) = \Sigma \mp a\chi, \quad \Phi^3_{\lambda^\pm}(U) = \Pi - a^2 d, \quad \Phi^4_{\lambda^\pm}(U) = \pm ah - \Pi. \end{aligned}$$

Let us note that the nonlinear system $(52)(53)_{\mu=0}$ is in non-conservation form since the product $(\Pi^2 - \chi)\partial_x Z$ never recasts in a divergence form. As a consequence, the Rankine-Hugoniot relations are here unknown. Now, let us recall that the Riemann invariants for a linearly degenerate field stay continuous across the associated contact wave. Since the system $(52)(53)_{\mu=0}$ solely involved linearly degenerate field, we deduce from the Riemann invariant formulas that the nonconservative product is never ambiguous (see [12] where ambiguities involved by product in non conservation form are considered, or [7] for example of non ambiguous nonconservative products). As a consequence, we can exhibit the Riemann solution for the nonconservative system $(52)(53)_{\mu=0}$. Let $U_\pm$ be constant states in $\mathcal{V}$. Consider the initial state U_0 defined as follows:

$$U_0(x) = \begin{cases} U_- & \text{if } x < 0, \\ U_+ & \text{if } x > 0. \end{cases} \tag{55}$$

The Riemann solution of $(52)(53)_{\mu=0}(55)$ consists in three contact discontinuities propagating at characteristic velocities:

$$U(x,t) = \begin{cases} U_- & \text{if } x/t < -a, \\ U_1 & \text{if } -a < x/t < 0, \\ U_2 & \text{if } 0 < x/t < a, \\ U_+ & \text{if } x/t > a, \end{cases} \tag{56}$$

where the intermediate states are evaluated involving the continuity of the Riemann invariants across their associated contact discontinuity waves. We obtain

$$\begin{aligned}
&Z_1 = Z_-, \quad Z_2 = Z_+,\\
&h_1 = h_2 = h^\star = \frac{h_- + h_+}{2} - \frac{1}{2a}(\Pi_+ - \Pi_-),\\
&\Pi_1 = \Pi_2 = \Pi^\star = \frac{\Pi_- + \Pi_+}{2} - \frac{a}{2}(h_+ - h_-),\\
&\chi_1 = \chi_2 = \chi^\star = \frac{a(\chi_- + \chi_+) - (\Sigma_+ - \Sigma_-) + (\Pi^\star)^2 \frac{Z_+ - Z_-}{\epsilon}}{2a + \frac{Z_+ - Z_-}{\epsilon}},\\
&d_1 = d_- - \frac{1}{a^2}(\Pi_- - \Pi^\star), \quad d_2 = d_+ + \frac{1}{a^2}(\Pi_+ - \Pi^\star),\\
&\Sigma_1 = \Sigma_- + a(\chi_- - \chi^\star), \quad \Sigma_2 = \Sigma_+ - a(\chi_+ - \chi^\star).
\end{aligned}$$

We conclude this brief analysis of the relaxation model (52)(53) by establishing the following positive preserving property satisfied by the Riemann solver under consideration:

Lemma 7. *Assume $U_\pm \in \mathcal{V}$ to be equilibrium state; i.e. $\Pi_\pm = \dfrac{d_\pm}{1+\chi_\pm}$ and $\Sigma_\pm = 0$. Assume $a > 0$ and $Z_+ - Z_- > 0$. Then $U(x,t)$, solution of the Riemann problem $(52)(53)_{\mu=0}$(55), stays entirely in $\mathcal{V}$.*

We omit the proof of the above result since it is a direct consequence of the formula of $\chi^\star$.

3.2. **A relaxation scheme.** We derive numerical approximations of the weak solutions of (50)(51) with the relaxation model (52)(53). At the very discrepancy with the splitting schemes, introduced in Section 2, the resulting relaxation scheme involves a relevant approximation of the source term. Such a *linearization* of the source term makes the scheme explicit and, sometime, reduces the cost of the numerical experiments.

To access such an issue, a two step method is adopted. The first step, called *evolution step*, is devoted to evolve the approximate solution from u_i^n to $u_i^{n+\frac{1}{2}}$ involving the relaxation model with $\mu = 0$. During the second step, the *relaxation step*, the relaxation source terms are approximated in the regime $\mu = \infty$.

First, assume known a piecewise constant approximation of the solution, u_i^n, at time t_n. Let us introduce an equilibrium piecewise constant state $U_i^n := {}^t(u_i^n, Z_i^n, \Pi_i^n = d_i^n/(1+\chi_i^n), \Sigma_i^n = 0)$. This equilibrium state is now evolved in time by a Godunov scheme for $(52)(53)_{\mu=0}$. Let us recall that the Godunov scheme for $(52)(53)_{\mu=0}$ is derived by solving elementary Riemann problems stated at each interface $x_{i+\frac{1}{2}}$ and integrating the obtain Riemann solutions, given by (56), on $(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$ at time $t_n + \Delta t$. Now, we have to fix the relaxation paramater a. At each interface $x_{i+\frac{1}{2}}$, we define the parameter $a_{i+\frac{1}{2}}$ according to the Whitham condition (54). Assuming the following CFL like restriction:

$$\frac{\Delta t}{\Delta x} \max_{i \in \mathbb{Z}} a_{i+\frac{1}{2}} \leq \frac{1}{2}, \tag{57}$$

tha relaxation parameter a may vary from one interface to another. Setting $Z_{i+1} - Z_i = \Delta x$, after computation we obtain

$$\left\{\begin{aligned} d_i^{n+\frac{1}{2}} &= d_i^n - \frac{\Delta t}{\Delta x}(h_{i+\frac{1}{2}}^n - h_{i-\frac{1}{2}}^n), \\ h_i^{n+\frac{1}{2}} &= h_i^n - \frac{\Delta t}{\Delta x}(\Pi_{i+\frac{1}{2}}^n - \Pi_{i-\frac{1}{2}}^n), \\ \chi_i^{n+\frac{1}{2}} &= \chi_i^n - \frac{\Delta t}{\Delta x}(\Sigma_{i+\frac{1}{2}}^n - \Sigma_{i-\frac{1}{2}}^n) + \Delta t S_i^n, \end{aligned}\right. \tag{58}$$

where the numerical Godunov flux functions are given as follows:

$$\left\{\begin{aligned} h_{i+\frac{1}{2}} &= \frac{h_i + h_{i+1}}{2} - \frac{1}{2a_{i+\frac{1}{2}}}(e_{i+1} - e_i), \\ \Pi_{i+\frac{1}{2}} &= \frac{e_i + e_{i+1}}{2} - \frac{a_{i+\frac{1}{2}}}{2}(h_{i+1} - h_i), \\ \Sigma_{i+\frac{1}{2}} &= -\epsilon a_{i+\frac{1}{2}} \alpha_{i+\frac{1}{2}}(\chi_{i+1} - \chi_i). \end{aligned}\right. \tag{59}$$

The source term is defined by

$$S_i = (\alpha_{i+\frac{1}{2}} + \alpha_{i-\frac{1}{2}}) \left(\frac{\alpha_{i+\frac{1}{2}} \Pi^2_{i+\frac{1}{2}} + \alpha_{i-\frac{1}{2}} \Pi^2_{i-\frac{1}{2}}}{\alpha_{i+\frac{1}{2}} + \alpha_{i-\frac{1}{2}}} - \chi_i^n \right), \tag{60}$$

$$\alpha_{i+\frac{1}{2}} = \frac{a_{i+\frac{1}{2}}}{2a_{i+\frac{1}{2}}\epsilon + \Delta x}. \tag{61}$$

The second step of the relaxation method consists in solving the following system:

$$\left\{\begin{aligned} &\partial_t d = 0, \\ &\partial_t h = 0, \\ &\partial_t \chi = 0, \\ &\partial_t Z = 0, \end{aligned}\right. \quad \text{and} \quad \left\{\begin{aligned} &\partial_t \Pi = \mu \left(\frac{d}{1+\chi} - \chi \right), \\ &\partial_t \Sigma = -\mu \Sigma, \end{aligned}\right.$$

with $U_i^{n+\frac{1}{2}}$ as initial data. Whenever μ tends to infinity, the final scheme reads as follows:

$$\left\{\begin{aligned} d_i^{n+1} &= d_i^{n+\frac{1}{2}}, \\ h_i^{n+1} &= h_i^{n+\frac{1}{2}}, \\ \chi_i^{n+1} &= \chi_i^{n+\frac{1}{2}}, \end{aligned}\right. \quad \text{and} \quad \left\{\begin{aligned} \Pi_i^{n+1} &= \frac{d_i^{n+1}}{1+\chi_i^{n+1}}, \\ \Sigma_i^{n+1} &= 0. \end{aligned}\right.$$

To conclude the derivation of the relaxation scheme, let us note that this numerical method is positive preserving since the associated approximate Riemann solver is positive preserving (see Lemma 7). As a consequence, we have $\chi_i^{n+1} \geq 0$ as soon as $\chi_i^n \geq 0$ for all i in $\mathbb{Z}$.

Concerning the asymptotic behavior of the method, it is clear that the standard asymptotic preserving property cannot be reached since, for $\epsilon = 0$, we have

$$\chi_i^{n+1} = \chi_i^n - \frac{\Delta t}{\alpha_{i+\frac{1}{2}} + \alpha_{i-\frac{1}{2}}} \left(\frac{\alpha_{i+\frac{1}{2}} (\Pi^n_{i+\frac{1}{2}})^2 + \alpha_{i-\frac{1}{2}} (\Pi^n_{i-\frac{1}{2}})^2}{\alpha_{i+\frac{1}{2}} + \alpha_{i-\frac{1}{2}}} - \chi_i^n \right).$$

In fact, the asymptotic behavior of the scheme cannot be analyzed independently of the convergence when Δx tends to zero. Indeed we have (similar idea can be found in [1]):

Lemma 8. *Assume the ratio $\frac{\Delta x}{\epsilon}$ admits a limit, denoted ℓ, as ϵ tends to zero. The evolution law for χ_i^{n+1}, in the limit of ϵ to zero, writes*

$$\chi_i^n = \frac{\beta_{i+\frac{1}{2}}(\Pi^n_{i+\frac{1}{2}})^2 + \beta_{i-\frac{1}{2}}(\Pi^n_{i-\frac{1}{2}})^2}{\beta_{i+\frac{1}{2}} + \beta_{i-\frac{1}{2}}}, \tag{62}$$

where $\beta_{i+\frac{1}{2}} > 0$ where $\beta_{i+\frac{1}{2}} = a_{i+\frac{1}{2}}/(2a_{i+\frac{1}{2}} + \ell)$.

Proof. Let the evolution law for χ_i^{n+1} be rewritten as follows:

$$\begin{aligned}
&\epsilon\chi_i^{n+1} = \epsilon\chi_i^n - \\
&\epsilon\frac{\Delta t}{\Delta x}\left(-\frac{a^2_{i+\frac{1}{2}}}{2a_{i+\frac{1}{2}} + \frac{\Delta x}{\epsilon}}(\chi^n_{i+1} - \chi^n_i) + \frac{a^2_{i-\frac{1}{2}}}{2a_{i-\frac{1}{2}} + \frac{\Delta x}{\epsilon}}(\chi^n_i - \chi^n_{i-1})\right) + \\
&\Delta t\left(\frac{a_{i+\frac{1}{2}}}{2a_{i+\frac{1}{2}} + \frac{\Delta x}{\epsilon}}(\Pi^n_{i+\frac{1}{2}})^2 + \frac{a_{i-\frac{1}{2}}}{2a_{i-\frac{1}{2}} + \frac{\Delta x}{\epsilon}}(\Pi^n_{i-\frac{1}{2}})^2 - \right.\\
&\qquad\qquad \left.\left(\frac{a_{i+\frac{1}{2}}}{2a_{i+\frac{1}{2}} + \frac{\Delta x}{\epsilon}} + \frac{a_{i-\frac{1}{2}}}{2a_{i-\frac{1}{2}} + \frac{\Delta x}{\epsilon}}\right)\chi^n_i\right).
\end{aligned}$$

The proof is completed as soon as ϵ tends to zero. □

Note that $\Pi^n_{i+\frac{1}{2}}$ is nothing but a discrete form of $\frac{d}{1+\chi}$ at the cell interface $x_{i+\frac{1}{2}}$. As a consequence, (62) turns out to be a discrete form of the expected equilibrium relation $\chi = (\frac{d}{1+\chi})^2$.

The numerical experiments performed with this method are in a good agreement with the expected solution. In addition, the method will be shown to be able to capture the asymptotic regime: better than expected by the theoritical result of Lemma 8, we can fix Δx and let ϵ go to zero. However, this scheme turns out to be poorly accurate. It does not capture the stationary contact discontinuities for χ and it introduces too much numerical viscosity. Actually this scheme is the starting point for a better numerical procedure, as described in the following Subsection.

3.3. **A Godunov extension.**

In the framework of the Kerr-Debye model, the Godunov method produces an accurate scheme. When considering such an approach, the main difficulty stays in the discretization of the source term. To access such an issue, we suggest to adopt the discrete source term, involved in (60), but for the Godunov characteristic speed. Put in other words, we adopt the discrete formula (60) where the characteristic velocities, $a_{i+\frac{1}{2}}$ and $a_{i-\frac{1}{2}}$, involved on the cell $(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$ are now defined by $\dfrac{1}{\sqrt{1+\chi_i^n}}$. As a consequence, the discrete source term reads as follows:

$$S_i = \frac{\frac{2}{\sqrt{1+\chi_i}}}{\frac{2\epsilon}{\sqrt{1+\chi_i}} + \Delta x}\left(\frac{e^2_{i+\frac{1}{2}} + e^2_{i-\frac{1}{2}}}{2} - \chi_i\right). \tag{63}$$

Thus we suggest the following scheme:

$$\left\{\begin{aligned}
&d_i^{n+1} = d_i^n - \frac{\Delta t}{\Delta x}(h^n_{i+\frac{1}{2}} - h^n_{i-\frac{1}{2}}),\\
&h_i^{n+1} = h_i^n - \frac{\Delta t}{\Delta x}(e^n_{i+\frac{1}{2}} - e^n_{i-\frac{1}{2}}),\\
&\chi_i^{n+1} = \chi_i^n + \Delta t S_i^n,
\end{aligned}\right. \tag{64}$$

where the numerical flux function $h_{i+\frac{1}{2}}$ and $e_{i+\frac{1}{2}}$ are the Godunov numerical flux (23), and the source term S_i is given by (63). Let us note that the scheme (64) is an explicit approximate solver, and its implementation turns out to be very easy.

Concerning the asymptotic behavior of the scheme (23)(63)(64) for $\epsilon \to 0$, we have

Lemma 9. *Assume the ratio $\Delta x/\epsilon$ admits a limit as ϵ tends to zero. In the limit of ϵ to zero, the sequence $(\chi_i^n)_{i\in\mathbb{Z}}$ satisfies:*

$$\chi_i^n = \frac{1}{2}((e_{i+\frac{1}{2}}^n)^2 + (e_{i-\frac{1}{2}}^n)^2). \tag{65}$$

Proof. From (63)(64), let us rewrite the evolution laws for χ_i^n as follows:

$$\epsilon\chi_i^{n+1} = \epsilon\chi_i^n + \Delta t \frac{\frac{2}{\sqrt{1+\chi_i}}}{\frac{2}{\sqrt{1+\chi_i}} + \frac{\Delta x}{\epsilon}} \left(\frac{e_{i+\frac{1}{2}}^2 + e_{i-\frac{1}{2}}^2}{2} - \chi_i \right).$$

The proof is achieved whenever ϵ tends to zero. □

Let us emphasize that $e_{i+\frac{1}{2}}$ is a discrete form at the cell interface $x_{i+\frac{1}{2}}$ of the continuous relation $e = d/(1+\chi)$. As a consequence, the above relation (65) is nothing but a discrete form of the equilibrium equation $\chi = (\frac{d}{1+\chi})^2$.

Now, we turn establishing the robustness and the stability properties satisfied by this numerical method. To access such an issue, let us first prove that the scheme (23)(63)(64) is a Godunov like scheme. Indeed, consider the Riemann solver (21) but for intermediate constant states defined as follows:

$$\begin{cases} h_1 = h_2 = h^\star = \dfrac{-(e_+ - e_-)\sqrt{1+\chi_-}\sqrt{1+\chi_+} + h_+\sqrt{1+\chi_-} + h_-\sqrt{1+\chi_+}}{\sqrt{1+\chi_+} + \sqrt{1+\chi_-}} \\ d_1 = d_- - (h^\star - h_-)\sqrt{1+\chi_-} \\ d_2 = d_+ - (h_+ - h^\star)\sqrt{1+\chi_+} \\ \chi_1 = \dfrac{2}{2 + \frac{\Delta x}{\epsilon}\sqrt{1+\chi_-}}\chi_- + \dfrac{\frac{\Delta x}{\epsilon}\sqrt{1+\chi_-}}{2 + \frac{\Delta x}{\epsilon}\sqrt{1+\chi_-}} \left(\dfrac{d_1}{1+\chi_-} \right)^2, \\ \chi_2 = \dfrac{2}{2 + \frac{\Delta x}{\epsilon}\sqrt{1+\chi_+}}\chi_+ + \dfrac{\frac{\Delta x}{\epsilon}\sqrt{1+\chi_+}}{2 + \frac{\Delta x}{\epsilon}\sqrt{1+\chi_+}} \left(\dfrac{d_2}{1+\chi_+} \right)^2. \end{cases} \tag{66}$$

The Godunov type scheme interpretation of the scheme (23)(63)(64) is thus established in the following result:

Lemma 10. *For $i \in \mathbb{Z}$ and $n \geq 0$, let $u_{h,i+\frac{1}{2}}^n(x,t)$ be the approximate Riemann solution (21)(66) stated at the cell interface $x_{i+\frac{1}{2}}$, where we have set $u_- = u_i^n$ and $u_+ = u_{i+1}^n$. Under the half CFL condition (26), define the function u_h by*

$$u_h(x,t) = u_{h,i+\frac{1}{2}}^n(x,t) \text{ if } x \in]x_i, x_{i+1}[.$$

Then, the updated state u_i^{n+1}, defined by (23)(63)(64), reads as follows:

$$u_i^{n+1} = \frac{1}{\Delta x}\int_{C_i} u_h(x, t_{n+1})dx. \tag{67}$$

Proof. This result is classical (see [17]) and we give the proof for the sake of completeness. Under the CFL restriction (26), since the juxtaposition of the approximate Riemann solutions (21)(66) are noninteracting, the expected result comes from a direct evaluation of the integral. Indeed, setting $\lambda = \frac{1}{\sqrt{1+\chi_i^n}}$, we have

$$\begin{aligned}
\frac{1}{\Delta x}\int_{C_i} u_h(x,t_{n+1})dx &= \int_{x_{i-\frac12}}^{x_{i-\frac12}+\lambda\Delta t} u^{n,2}_{h,i-\frac12}dx \\
&\quad + \frac{1}{\Delta x}\int_{x_{i-\frac12}+\lambda\Delta t}^{x_{i+\frac12}-\lambda\Delta t} u_i^n dx + \frac{1}{\Delta x}\int_{x_{i+\frac12}-\lambda\Delta t}^{x_{i+\frac12}} u^{n,1}_{h,i+\frac12}dx, \\
&= u_i^n - \lambda\frac{\Delta t}{\Delta x}\left((u_i^n - u^{n,2}_{h,i-\frac12}) + (u_i^n - u^{n,1}_{h,i+\frac12})\right).
\end{aligned}$$

Involving the definition of $u^{n,1}_{h,i+\frac12}$ and $u^{n,2}_{h,i-\frac12}$, deduced from (66), a straightforward computation gives the expected identity (67) and the proof is thus completed. □

Since it will be usefull in the sequel, involving the intermediate states (66), let us note that the Godunov numerical flux function $e_{i+\frac12}$ now reads

$$(68) \qquad e^n_{i+\frac12} = \frac{d^{n,1}_{i+\frac12}}{1+\chi_i^n} = \frac{d^{n,2}_{i+\frac12}}{1+\chi^n_{i+1}}.$$

Now, we give the main statement concerning the robustness and the stability of the Godunov type scheme (23)(63)(64).

Theorem 2. *Let $\chi_i^n \geq 0$ for all i in $\mathbb{Z}$. Assume the CFL condition (26) and consider the updated state u_i^{n+1} given by the numerical scheme (23)(63)(64). Then $\chi_i^{n+1} \geq 0$ for all i in $\mathbb{Z}$. In addition, as soon as Δx is small enough and $\epsilon > 0$ is fixed, the following discrete entropy inequality holds:*

$$\frac{\mathcal{E}_{KD}(u_i^{n+1}) - \mathcal{E}_{KD}(u_i^n)}{\Delta t} + \frac{h_{i+\frac12}e_{i+\frac12} - h_{i-\frac12}e_{i-\frac12}}{\Delta x} \leq 0,$$

where the Kerr-Debye entropy function $\mathcal{E}_{KD}$ is defined by (10).

Proof. First, let us adopt the notations introduced Lemma 10. From (66), we note that $\chi_1 \geq 0$ and $\chi_2 \geq 0$ as soon as $\chi_\pm \geq 0$. As a consequence, with $\chi_i^n \geq 0$ for all i in $\mathbb{Z}$, the function $\chi_h(x,t)$ is nonnegative. Now, involving the integral definition of χ_i^{n+1}, given by (67), we immediately deduce that $\chi_i^{n+1} \geq 0$ for all i in $\mathbb{Z}$.

Next, concerning the discrete entropy inequality, by Lemma 10 and the Jensen's inequality we have

$$\mathcal{E}_{KD}(u_i^{n+1}) \leq \mathcal{E}_{KD}(u_i^n) - \frac{\Delta t}{\Delta x\sqrt{1+\chi_i^n}}\left(-\mathcal{E}_{KD}(u^{n,2}_{h,i-\frac12}) + 2\mathcal{E}_{KD}(u_i^n) - \mathcal{E}_{KD}(u^{n,1}_{h,i+\frac12})\right).$$

For the sake of simplicity in the notations, let us introduce the function $\phi \in C^1(\mathbb{R}_+,\mathbb{R})$, defined as follows:

$$\begin{aligned}
\phi(\Delta x) &= -\mathcal{E}_{KD}(u^{n,2}_{h,i-\frac12}) + 2\mathcal{E}_{KD}(u_i^n) - \mathcal{E}_{KD}(u^{n,1}_{h,i+\frac12}), \\
&= 2\left(\frac{(d_i^n)^2}{2(1+\chi_i^n)} + \frac{(h_i^n)^2}{2} + \frac{(\chi_i^n)^2}{4}\right) - \\
&\quad \left(\frac{(d^{n,2}_{i-\frac12})^2}{2(1+\chi^{n,2}_{i-\frac12})} + \frac{(h^{n,2}_{i-\frac12})^2}{2} + \frac{(\chi^{n,2}_{i-\frac12})^2}{4}\right) - \\
&\quad \left(\frac{(d^{n,1}_{i+\frac12})^2}{2(1+\chi^{n,1}_{i+\frac12})} + \frac{(h^{n,1}_{i+\frac12})^2}{2} + \frac{(\chi^{n,1}_{i+\frac12})^2}{4}\right),
\end{aligned}$$

to write

$$\mathcal{E}_{KD}(u_i^{n+1}) \leq \mathcal{E}_{KD}(u_i^n) - \frac{\Delta t}{\Delta x \sqrt{1+\chi_i^n}} \phi(\Delta x).$$

From (66) to define $u_{h,i-\frac{1}{2}}^{n,2}$ and $u_{h,i+\frac{1}{2}}^{n,1}$, note that the dependence on Δx of the function ϕ just comes from the definition of $\chi_{i-\frac{1}{2}}^{n,2} := \chi_{i-\frac{1}{2}}^{n,2}(\Delta x)$ and $\chi_{i+\frac{1}{2}}^{n,1} := \chi_{i+\frac{1}{2}}^{n,1}(\Delta x)$. With $\Delta x = 0$, we have $\chi_{i-\frac{1}{2}}^{n,2}(\Delta x = 0) = \chi_i^n$ and $\chi_{i+\frac{1}{2}}^{n,1}(\Delta x = 0) = \chi_i^n$ to write

$$\phi(0) = \sqrt{1+\chi_i^n}\left(\frac{1}{\sqrt{1+\chi_i^n}}\left(\mathcal{E}_{KD}(u_i^n) - \mathcal{E}_{KD}(w_{i+\frac{1}{2}}^{n,1})\right) + \frac{1}{\sqrt{1+\chi_i^n}}\left(\mathcal{E}_{KD}(u_i^n) - \mathcal{E}_{KD}(w_{i-\frac{1}{2}}^{n,2})\right)\right),$$

where we have set

$$w_{i+\frac{1}{2}}^{n,1} = \left(d_{i+\frac{1}{2}}^{n,1}, h_{i+\frac{1}{2}}^n, \chi_i^n\right) \quad \text{and} \quad w_{i-\frac{1}{2}}^{n,2} = \left(d_{i-\frac{1}{2}}^{n,2}, h_{i-\frac{1}{2}}^n, \chi_i^n\right).$$

Now, Lemma 5 can be applied to write

$$\frac{1}{\sqrt{1+\chi_i^n}}(\mathcal{E}_{KD}(u_i^n) - \mathcal{E}_{KD}(w_{i+\frac{1}{2}}^{n,1})) = h_{i+\frac{1}{2}}^n e_{i+\frac{1}{2}}^n - h_i^n e_i^n,$$

$$\frac{1}{\sqrt{1+\chi_i^n}}(\mathcal{E}_{KD}(u_i^n) - \mathcal{E}_{KD}(w_{i-\frac{1}{2}}^{n,2})) = h_i^n e_i^n - h_{i-\frac{1}{2}}^n e_{i-\frac{1}{2}}^n.$$

As a consequence, $\phi(0)$ reads as follows:

$$\phi(0) = \sqrt{1+\chi_i^n}\left(h_{i+\frac{1}{2}}^n e_{i+\frac{1}{2}}^n - h_{i-\frac{1}{2}}^n e_{i-\frac{1}{2}}^n\right).$$

Next, let us analyze the behavior of the function ϕ in a neighborhood of zero. Concerning the variation of ϕ we have

$$\begin{aligned} \phi'(\Delta x) &= \frac{1}{2}\left(\left(\frac{d_{i-\frac{1}{2}}^{n,2}}{1+\chi_{i-\frac{1}{2}}^{n,2}(\Delta x)}\right)^2 - \chi_{i-\frac{1}{2}}^{n,2}(\Delta x)\right)\frac{d}{d\Delta x}\chi_{i-\frac{1}{2}}^{n,2}(\Delta x) + \\ &\quad \frac{1}{2}\left(\left(\frac{d_{i+\frac{1}{2}}^{n,1}}{1+\chi_{i+\frac{1}{2}}^{n,1}(\Delta x)}\right)^2 - \chi_{i+\frac{1}{2}}^{n,1}(\Delta x)\right)\frac{d}{d\Delta x}\chi_{i+\frac{1}{2}}^{n,1}(\Delta x), \end{aligned}$$

where

$$\frac{d}{d\Delta x}\chi_{i-\frac{1}{2}}^{n,2}(\Delta x) = \frac{\frac{2\epsilon}{\sqrt{1+\chi_i^n}}}{\left(\frac{2\epsilon}{\sqrt{1+\chi_i^n}} + \Delta x\right)^2}\left(\left(\frac{d_{i-\frac{1}{2}}^{n,2}}{1+\chi_i^n}\right)^2 - \chi_i^n\right),$$

$$\frac{d}{d\Delta x}\chi_{i+\frac{1}{2}}^{n,1}(\Delta x) = \frac{\frac{2\epsilon}{\sqrt{1+\chi_i^n}}}{\left(\frac{2\epsilon}{\sqrt{1+\chi_i^n}} + \Delta x\right)^2}\left(\left(\frac{d_{i+\frac{1}{2}}^{n,1}}{1+\chi_i^n}\right)^2 - \chi_i^n\right).$$

Since we have $\chi_{i-\frac{1}{2}}^{n,2}(0) = \chi_i^n$ and $\chi_{i+\frac{1}{2}}^{n,1}(0) = \chi_i^n$, we immediately deduce

$$\phi'(0) = \frac{\sqrt{1+\chi_i^n}}{4\epsilon}\left(\left(\left(\frac{d_{i-\frac{1}{2}}^{n,2}}{1+\chi_i^n}\right)^2 - \chi_i^n\right)^2 + \left(\left(\frac{d_{i+\frac{1}{2}}^{n,1}}{1+\chi_i^n}\right)^2 - \chi_i^n\right)^2\right) \geq 0.$$

If $\phi'(0) > 0$, invoking standard continuity arguments, we obtain $\phi(\Delta x) > \phi(0)$ as soon as Δx is small enough. Now, with $\phi'(0) = 0$ we easily obtain

$$\chi_i^n = \left(\frac{d_{i-\frac{1}{2}}^{n,2}}{1+\chi_i^n}\right)^2 = (e_{i-\frac{1}{2}}^n)^2 \quad \text{and} \quad \chi_i^n = \left(\frac{d_{i+\frac{1}{2}}^{n,1}}{1+\chi_i^n}\right)^2 = (e_{i+\frac{1}{2}}^n)^2.$$

As a consequence of the definition of the Godunov flux function (68), the source term S_i^n, defined by (63), vanishes independently of Δx. Then we obtain $\phi(\Delta x) = \phi(0)$ for all $\Delta x \geq 0$. We have thus proved that $\phi(\Delta x) \geq \phi(0)$ for all $\chi_i^n \geq 0$ and Δx small enough.

To conclude, by definition of ϕ we have for all $\Delta x > 0$ small enough

$$\mathcal{E}_{KD}(u_i^{n+1}) \leq \mathcal{E}_{KD}(u_i^n) - \frac{\Delta t}{\Delta x\sqrt{1+\chi_i^n}}\phi(0).$$

The proof is thus achieved. □

To conclude the derivation of the Godunov type scheme (23)(63)(64), let us emphasize that the above result solely establishes a discrete entropy inequality while one wants to specify this inequality according to the entropy dissipation relation (12). It is possible to exhibit a discrete entropy dissipation with Δx small enough. Indeed, as the function ϕ is smooth enough, we can perform an asymptotic expansion of ϕ in a neighborhood of zero, to write:

$$\mathcal{E}_{KD}(u_i^{n+1}) \leq \mathcal{E}_{KD}(u_i^n) - \frac{\Delta t}{\Delta x\sqrt{1+\chi_i^n}}\left(\phi(0) + \Delta x\phi'(0) + \mathcal{O}(\Delta x^2)\right).$$

From the obtained evaluations of $\phi(0)$ and $\phi'(0)$, the above inequality reads:

$$\frac{\mathcal{E}_{KD}(u_i^{n+1}) - \mathcal{E}_{KD}(u_i^n)}{\Delta t} + \frac{h_{i+\frac{1}{2}}^n e_{i+\frac{1}{2}}^n - h_{i-\frac{1}{2}}^n e_{i-\frac{1}{2}}^n}{\Delta x} \leq -\frac{1}{2\epsilon}\bar{S}_i^2 + \mathcal{O}(\Delta x^2),$$

where

$$\bar{S}_i^2 = \frac{1}{2}\left(\left(\frac{d_{i-\frac{1}{2}}^{n,2}}{1+\chi_i^n}\right)^2 - \chi_i^n\right)^2 + \frac{1}{2}\left(\left(\frac{d_{i+\frac{1}{2}}^{n,1}}{1+\chi_i^n}\right)^2 - \chi_i^n\right)^2,$$

to obtain a discrete form of (12).

4. Numerical experiments

In this section, we present numerical experiments performed with the schemes derived in the present paper.

We first present computations on Kerr-Debye relaxation shock profiles. Such functions are exact known solutions of Kerr-Debye system. Those tests validate the schemes when the positive parameter ϵ is fixed. All the schemes under consideration give good results.

Then a particular attention is paid to the asymptotic behavior of the solutions as ϵ tends to zero. The explicit scheme (Sect. 2.1) is shown converging to wrong solutions while the more sophisticate schemes: the implicit scheme (Sect. 2.3), the exact solving source term scheme (ESST, Sect. 2.5), the well-balanced relaxation scheme (WBR, Sect. 3.2) and the well-balanced modified Godunov scheme (WBMG, Sect. 3.3), give the expected solutions.

All the simulations are performed with an uniform mesh. As prescribed by the analysis, the CFL number is fixed to 0.5. Moreover, a second-order extension is proposed involving a standard MUSCL scheme (see van Leer [19]) based on a usual minmod reconstruction. We set the CFL number to 0.25 when the second-order scheme is used. The obtained numerical results are compared to the exact solutions given in [3].

The two first simulations are devoted to approximate relaxation shock profiles, that are smooth solutions of Kerr-Debye system (5) under the form

$$u_\epsilon(x,t) = W\left(\frac{x-\sigma t}{\epsilon}\right), \qquad W = (D, H, \Upsilon), \tag{69}$$

and such that

$$\begin{cases} D(\pm\infty) = d_\pm \\ H(\pm\infty) = h_\pm \\ \Upsilon(\pm\infty) = \chi_\pm \,. \end{cases} \tag{70}$$

In [3], it is proved that non trivial relaxation shock profiles exist if and only if

$$d_- \neq d_+ \quad d_-\, d_+ > 0, \quad \chi_\pm = p(d_\pm)^2, \tag{71}$$

and (d_-, h_-), (d_+, h_+) are connected by an entropic Kerr shock. Moreover, the profile is determined by the solution of an ODE which we solve numerically. We consider here one-shock solutions of the Kerr model with $d_- > d_+ > 0$ and we perform the computations with $\epsilon = 1$.

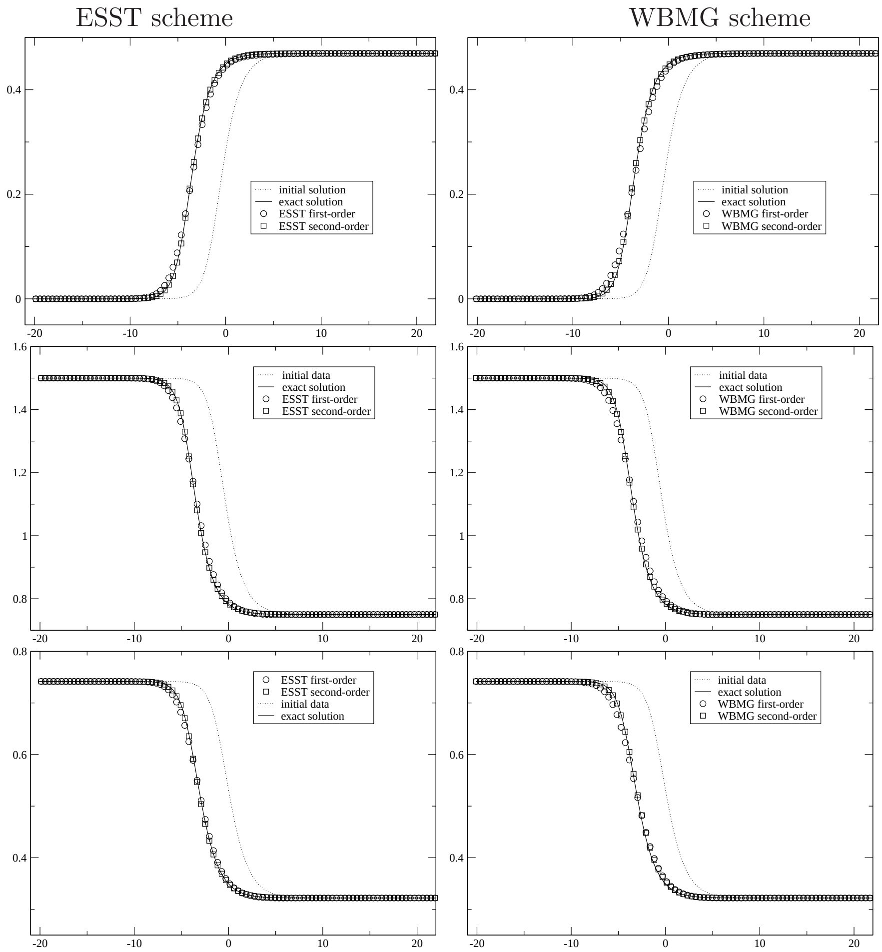

FIGURE 2. Relaxation shock profile 1: d, h and χ with first and second order schemes ESST and WBMG with an uniform mesh made of 100 cells

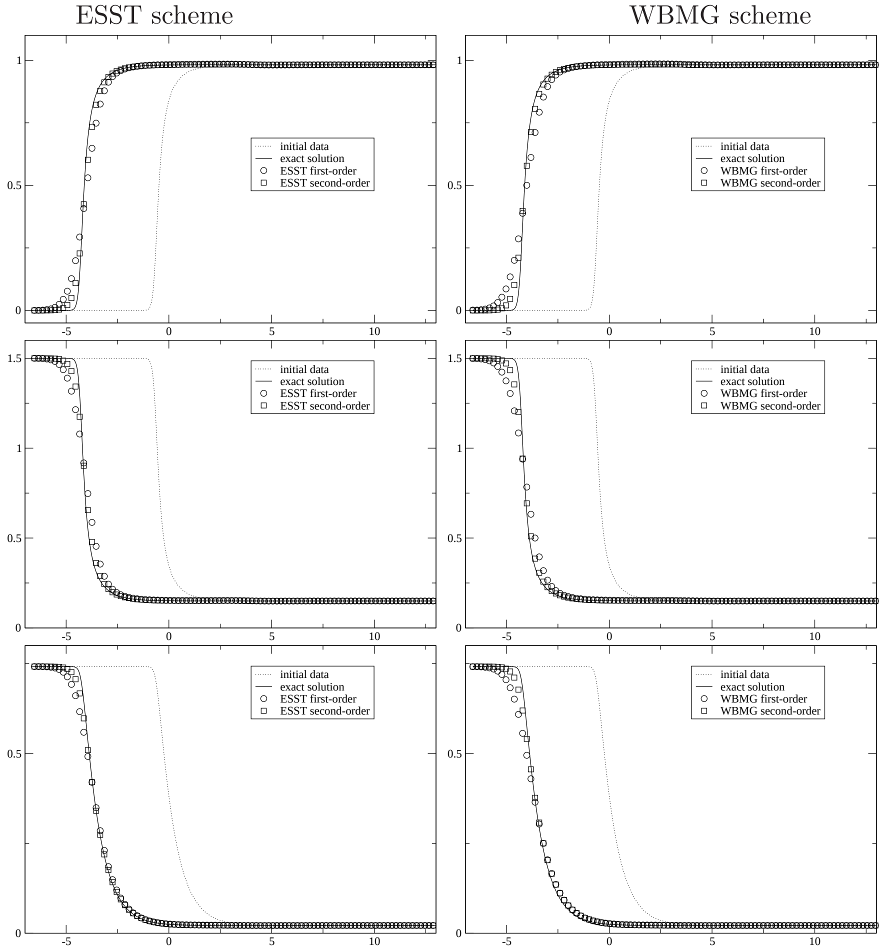

FIGURE 3. Relaxation shock profile 2: d, h and χ with first and second order schemes ESST and WBMG with an uniform mesh made of 100 cells

At time $t = 0$, the initial data is given by an exact relaxation profile. The first considered relaxation profile is characterized by $d_- = 1.5$ and $d_+ = 0.75$. It stays very smooth in time, see Figure 2. According to the theory [3], when d_+ tends to zero the profiles tend to a discontinuous solution of Kerr-Debye system. This behavior can be observed with our second profile, determined by $d_- = 1.5$ and $d_+ = 0.15$. At time $t = 5$, both approximated profile are displayed Figures 2 and 3 when considering the ESST and WBMG schemes. The comparison with the exact solution shows that the approximations are accurate. In fact, all the derived schemes in the present paper are able to capture those solutions as proved by the following L^1-error computations obtained involving the second-order extensions of the schemes:

First profile

cells	Explicit	Implicit	ESST	WBR	WBMG
100	5.46E-4	6.16E-4	5.67E-4	1.66E-3	1.46E-3
400	1.35E-4	7.73E-5	1.01E-4	3.06E-4	2.93E-4
800	7.25E-5	3.72E-5	5.60E-5	1.44E-4	1.41E-4
1600	3.75E-5	1.91E-5	3.24E-5	7.05E-5	6.97E-5

Second profile

cells	Explicit	Implicit	ESST	WBR	WBMG
100	1.38E-2	1.14E-2	1.13E-2	1.55E-2	1.35E-2
400	2.35E-3	1.22E-3	1.20E-3	2.35E-3	2.05E-3
800	1.20E-3	2.85E-4	2.76E-4	8.07E-4	7.21E-4
1600	3.76E-4	8.34E-5	8.71E-5	3.07E-4	2.86E-4

As expected, the WBMG scheme gives better results than the WBR one. We also remark that the implicit schemes are more accurate than the explicit ones. Among explicit approximations the splitting scheme is better than the well-balanced ones for the first (smooth) profile, while it becomes worse for the second (stiff) one. We underline the fact that the computationnal times are of the same order, except the one of the ESST scheme, which is greater.

Let us now turn our attention to the asymptotic behavior of the schemes, that is their behavior when ϵ tends to zero. It is expected that in this limit we obtain numerical approximations of the relaxed Kerr system. We fix the parameter $\epsilon = 0$. We recall that the limit of both implicit and ESST schemes coincide. We have verified that this coincidence is actually true numerically and we just present here the results for ESST, WBR and WBMG schemes. We consider an initial data made of two constant states separated by a discontinuity located at $x = 0$. This initial data is given as follows:

$$(d, h, \chi)(x, 0) = \begin{cases} (1.5, 0, 5) & \text{if } x < 0, \\ (-3, 1.5339, 5) & \text{if } x > 0. \end{cases}$$

We underline the fact that the initial value of χ has been fixed to a constant, $\chi(x, 0) = 5$, far from its equilibrium value given by $\chi^{eq}(x, 0) = p(d(x, 0))$. This is not important for the implicit splitting schemes because for $n \geq 1$ χ is automatically at equilibrium but it is for the well balanced approximations.

The exact solution of Kerr Riemann problem is known [3] and it is made of a composite wave (1-shock, 1-rarefaction) and a 2-shock wave. The ability of the schemes to capture the relevant Kerr regime is now tested.

In Figure 4, the numerical approximation obtained with the ESST scheme and both WBR and WBMG schemes are displayed at time $t = 1$. For the sake of clarity in the presentation of the numerical results, the approximations are centered in the interval of computation $(-2.0039, 1.2113)$. The results clearly show a very good agreement with the exact solution. This agreement is checked in the following L^1-error evaluations:

First-order schemes

cells	ESST	WBR	WBMG
100	5.03E-2	6.42E-2	6.23E-2
500	1.69E-2	1.86E-2	1.81E-2
1000	1.01E-2	1.12E-2	1.09E-2

Second-order schemes

cells	ESST	WBR	WBMG
100	2.82E-2	4.05E-2	3.81E-2
500	7.74E-3	8.80E-3	8.23E-3
1000	4.35E-3	4.96E-3	4.65E-3

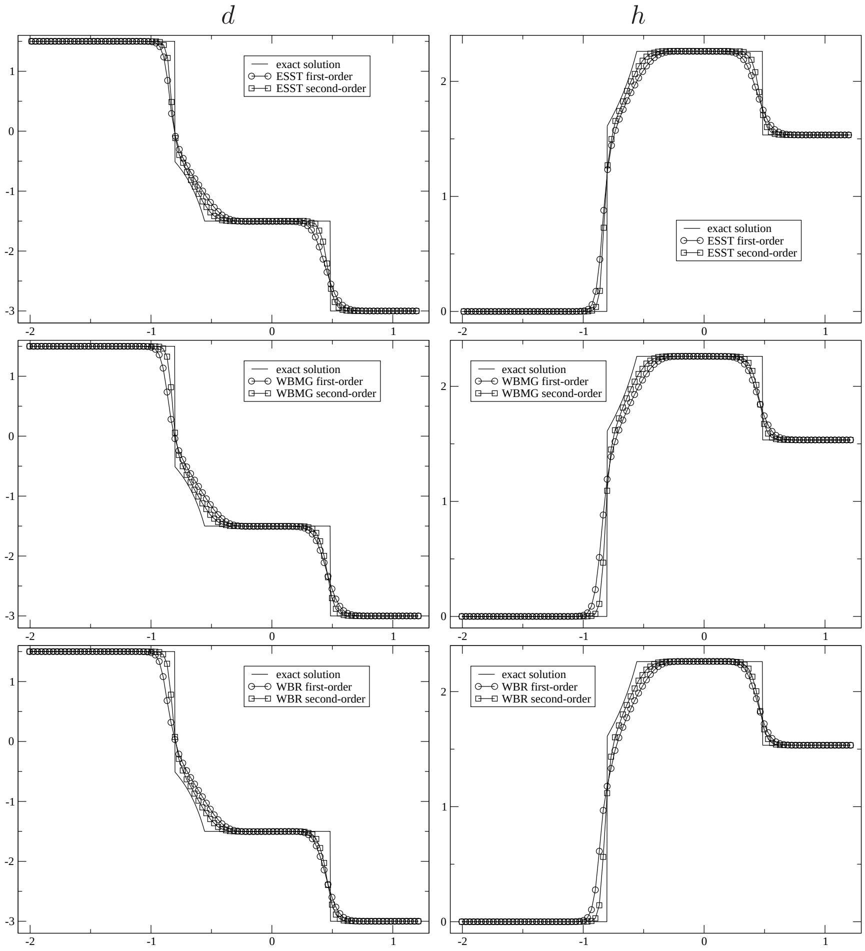

FIGURE 4. Riemann problem 1: d and h with first and second order schemes ESST, WBR and WBMG on the interval $(-2.0039, 1.2113)$ with an uniform mesh made of 100 cells

In Figure 5, we display the approximate solutions obtained with the explicit splitting schemes of Sect. 2.2. These schemes do not capture the exact solution. They do not own the correct limit.

The second Riemann problem is concerned with initial data as follows:

$$(d,h,\chi)(x,0) = \begin{cases} (1.5, 0, 5) & \text{if } x < 0, \\ (2.5958, 5.1153, 5) & \text{if } x > 0, \end{cases}$$

The exact solution is made of two composite waves (1-shock, 1-rarefaction and 2-rarefaction, 2-shock). In Figure 6, we exhibit the numerical solutions obtained with the schemes ESST (or equivalently the implicit scheme), WBR and WBMG. The approximations are displayed at time $t = 1$ on the interval $(-2.0039, 1.7831)$ to center the solution. The following L^1-error evaluations ensure the expected convergence of the three schemes:

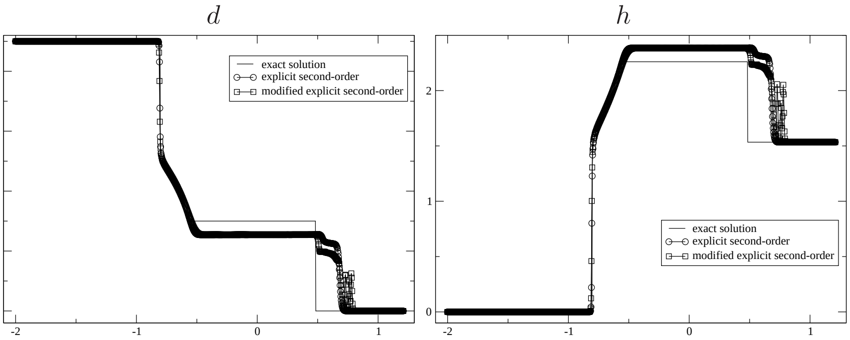

FIGURE 5. Riemann problem 1: d and h with the second order explicit splitting schemes on the interval $(-2.0039, 1.2113)$ with an uniform mesh made of 1000 cells

First-order schemes

cells	ESST	WBR	WBMG
100	5.08E-2	6.51E-2	5.77E-2
500	1.73E-2	2.14E-2	1.98E-2
1000	1.04E-2	1.28E-2	1.20E-2

Second-order schemes

cells	ESST	WBR	WBMG
100	3.28E-2	3.29E-2	3.09E-2
500	9.47E-3	9.81E-3	9.20E-3
1000	5.37E-3	5.59E-3	5.29E-3

5. Conclusion

In the present paper, we have derived numerical schemes to approximate the solutions of the Kerr-Debye model. The main difficulty of these equations comes from the approximation of the stiff source term. We have shown that basic explicit numerical procedures are relevant to approximate shock profiles but converge to wrong solutions in the Kerr relaxation limit. To overcome such a problem, we have introduced more sophisticate schemes: implicit splitting schemes and explicit well-balanced approximate Riemann solvers. These methods are proved to be positive and entropy preserving. The numerical experiments show their accuracy and stability for fixed positive ϵ as well as in the asymptotic Kerr regime.

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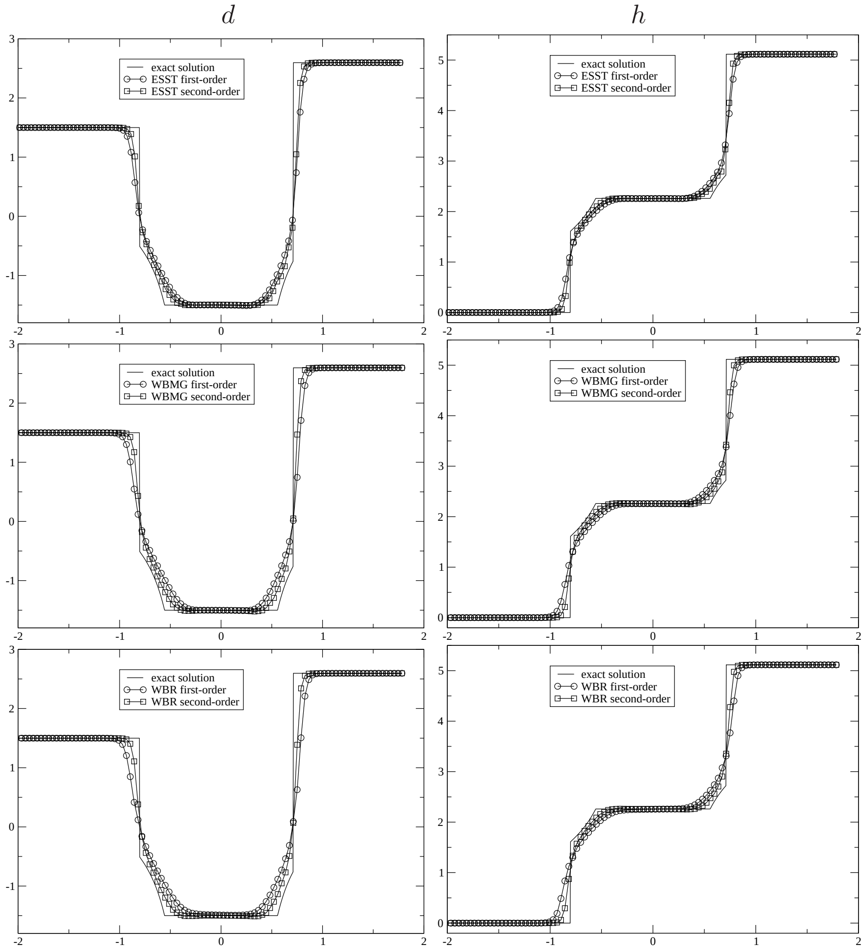

FIGURE 6. Riemann problem 2: d and h with first and second order schemes ESST, WBR and WBMG on the interval $(-2.0039, 1.7831)$ with an uniform mesh made of 100 cells

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